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Low-frequency approximations in unsteady small perturbation subsonic flows

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A more rigorous proof is given of the validity of a generalized Prandtl–Glauert technique for caculating the solution to time-dependent small perturbation flows. The method, first used by Miles (1950*a*) for the airfoil problem and later applied by Amiet & Sears (1970) to more general problems, allows calculation of the term of first order in frequency. Anomalous behaviour for the two-dimensional problem is examined in detail and found to be limited to those two-dimensional cases which include shed vorticity downstream of the body. This anomaly, which precludes using the method for these cases, results from the need to satisfy a velocity boundary condition on the body. For this purpose the velocity must be calculated from the basic variable, the pressure, through an integrated form of the momentum equation. It is in thus calculating the velocity that the anomaly occurs. The method can be applied to both the two-dimensional case without shed vorticity and the general three-dimensional case.

1. Introduction

For small perturbation subsonic flow past a body with a time-dependent boundary condition, such as an oscillating airfoil, it is possible to extend the familiar Prandtl–Glauert rule to include, in addition to the quasi-steady term, the term of first order in the frequency of oscillation. To be more precise, this term is $O(\epsilon)$, where $\epsilon \equiv M(\omega b/U)/(1-M^2) \equiv Mk^*$, ω being the circular frequency, b a typical body dimension, U the mean flow velocity, M the Mach number, and k^* the reduced frequency k divided by $1-M^2$. This technique involves an application of Galilean and Lorentz transformations to reduce the convective wave equation (the wave equation for a fluid with a mean flow) to the standard wave equation for zero mean flow. The boundary conditions must also be transformed, but the particular combination of transformations used leaves the body at rest (in contrast to a Galilean transform alone, which cannot bring both the body and the fluid to rest). By then ignoring the second-order time derivative in the wave equation, the problem can be reduced to Laplace's equation while ignoring only terms $O(\epsilon^2)$ and higher; i.e. the $O(\epsilon)$ term is retained. This solution technique is discussed in the paper by Amiet & Sears (1970) and also by Miles (1950*a*). It has been referred to as the GASP method (Sears 1971).

A curious anomaly occurs in this solution procedure, however, for the case of a two-dimensional (infinite-span) airfoil with shed vorticity downstream. For this particular case, ignoring the second-order time derivative results in a solution which has only the quasi-steady term correct, the term $O(\epsilon)$ being given incorrectly. This effect was first noted by Miles (1950*b*), who derived a solution correct to $O(k^*)$ directly from the Possio integral equation. A modification by Amiet (1974, 1975*a*) of this two-dimensional solution reduces the error from $O(k^{*2})$ to $O(\epsilon^2)$ and gives accurate results over a wide range of Mach number and reduced frequency when compared with numerical solutions, as is further verified by Kemp & Homicz (1976).

The reason given by Miles (1950*a*) for the anomaly is that the integration over the infinite span does not commute with the assumed expansion in powers of the frequency. While undoubtedly correct, this statement is difficult to interpret in a physical sense, and has left some readers with uncertainty as to the validity of the general technique. The intent of the present paper is to clarify the reason for the anomalous behaviour, and to verify, more rigorously than has been done before, that ignoring the second-order time derivative leads to correct $O(\epsilon)$ results for the general three-dimensional case and for the two-dimensional case without shed vorticity; e.g. a pulsating non-lifting airfoil for which there is no velocity discontinuity downstream.

Before proceeding to the detailed mathematical derivations, it is worth while to give a physical justification for the anomaly. The derivation of the Prandtl–Glauert extension is dependent on the assumption that the acoustic wavelength is much greater than a typical body dimension ($\epsilon \ll 1$). However, the shed vorticity, which influences the airfoil loading, itself has a length scale, namely its wavelength. Whereas the ratio of the body dimension to the acoustic wavelength goes to zero as the frequency goes to zero (the mean velocity remaining fixed), the ratio of the shed-vorticity wavelength to the acoustic wavelength remains fixed. Also, in the limit of small frequency, for the two-dimensional case it can be shown that the shed vorticity lying downstream of any fixed point influences the airfoil solution to first order in frequency. On the other hand, for the three-dimensional case the effect of the shed vorticity dies off more rapidly with distance from the airfoil, so that the far wake is of less importance than for the two-dimensional case.

There is a significant body of literature on singular perturbation theory (Van Dyke 1964) that points out the difficulties often encountered in neglecting one of the highest-order derivatives in an equation. For the present problem the neglect of the second-order time derivative transforms the wave equation (which is hyperbolic) into Laplace's equation (which is elliptic). This elliptic equation cannot be used to calculate the far-field solution (Amiet & Sears 1970). However, the far-field solution enters implicitly in the calculation of the near-field solution when the velocity is calculated from the pressure field in order to satisfy the boundary condition on velocity; i.e. see equation (4) herein.

Approximate solutions, such as those found by the methods described here, can be quite useful, especially for aero-acoustics problems. Although alternative numerical techniques for the solution of these problems usually exist, the improved accuracy obtained with such numerical solutions is not essential for noise calculations, and it is usually achieved at the expense of greater computing time. A case illustrating the need for reasonably simple solutions is that of noise generation by an airfoil interacting with a turbulent flow (see, for example, Amiet 1975*b*). Since the noise generated is broad band, many repetitions of the airfoil response calculation are needed in order to obtain a reasonable plot of the noise spectrum.

Although the methods discussed herein are limited to small ϵ, closed-form approximate solutions can often be derived for the large ϵ case also (e.g. Amiet 1976*a*). Finally, it should be emphasized that the methods described herein are not limited to airfoil problems, but can be applied to unsteady compressible potential flow problems in general.

2. Two-dimensional case with shed vorticity

To clarify the reason for the anomaly encountered in the two-dimensional case, the exact solutions for the pressure and velocity produced by a dipole normal to the flow will be compared with the approximate solutions found by neglecting the second-order time derivative. The equation for the pressure field produced by an unsteady line force

$$\mathbf{F} = \mathbf{j} F \delta(x)\, \delta(y)\, e^{i\omega t} \tag{1}$$

situated at the origin and normal to the mean flow ($\mathbf{j}$ is a unit vector in the y direction) is the convective wave equation

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{D^2}{Dt^2}\right) P = \nabla . \mathbf{F}, \tag{2}$$

where P is the perturbation pressure and $D/Dt \equiv \partial/\partial t + U\, \partial/\partial x$ is the substantial derivative. The solution of (2) is the pressure field of a two-dimensional dipole

$$P = -\frac{i\omega y F}{4\beta c_0 \sigma} H_1^{(2)}(\epsilon\bar{\sigma}) \exp\left[i(\omega t + \epsilon M \bar{x})\right], \tag{3}$$

where $\sigma^2 \equiv x^2 + \beta^2 y^2$, $\beta^2 \equiv 1 - M^2$ and c_0 and $H_n^{(2)}$ are respectively the sound speed and Hankel function of order n. An overbar indicates a variable non-dimensionalized by the body scale b.

The pressure field for an airfoil problem can be represented by a distribution of these dipoles over the airfoil surface. The velocity potential, because of the discontinuity across the wake, cannot be represented by a distribution of singular solutions such as these over the body surface alone, in contrast to the case when there is no shed vorticity downstream. (See also § 4.)

Because it is necessary to satisfy a velocity boundary condition on the airfoil surface, the velocity perturbation produced by the dipole pressure field must be calculated. This can be done by taking the y derivative of the velocity potential given by the following integrated form of the momentum equation:

$$\phi(x, y, t) = -\frac{1}{\rho_0 U}\int_{-\infty}^{x} P\left(\xi, y, t - \frac{x-\xi}{U}\right) d\xi. \tag{4}$$

This relation is commonly used in lifting-surface theory (e.g. Kussner 1940). Because the integral is taken from $-\infty$ to x, it is not too surprising that difficulties can be encountered when terms $O(\epsilon^2)$ are neglected in calculating the pressure, since the solution resulting from this assumption is incorrect at large distances. Introducing the pressure field given by (3), the velocity perturbation v normal to the mean flow on the x axis is found to be (see, for example, Fung 1969, p. 427)

$$v(x,0,t) = -\frac{\beta F I_1}{b\rho_0 U}\exp\left[i(\omega t - k\bar{x})\right], \tag{5}$$

where

$$I_1 = \frac{i\beta}{2\pi}k^*\ln\left(\frac{1+\beta}{M}\right) + \frac{k^*}{4}\left[iM\frac{x}{|x|}H_1^{(2)}(\epsilon|\bar{x}|) - H_0^{(2)}(\epsilon|\bar{x}|)\right]\exp(ik^*\bar{x}) + \frac{i}{4}\bar{x}(k^{*2}-\epsilon^2)\int_0^1 \exp(ik^*\bar{x}\xi)H_0^{(2)}(\epsilon|\bar{x}|\xi)\,d\xi. \tag{6}$$

It will be noted that, as well as representing the velocity on the x axis produced by a line force, the function v, with I_1 given by (6), is also the kernel of the Possio integral equation.

The exact dipole solution will be compared with the approximate solution found by transforming (2) to an X, Y, T co-ordinate system and neglecting the P_{TT} term, which is $O(\epsilon^2)$, in the resulting equation. The combination of Lorentz and Galilean transforms

$$x \to \beta X, \quad y \to Y, \quad t \to (T - MX/c_0)/\beta \tag{7}$$

used by Amiet & Sears (1970) and previously by other authors (e.g. Kussner 1940; Miles 1959) converts (2) into the standard wave equation. Neglecting the P_{TT} term, solving the resulting Laplace equation, and transforming back to the x, y, t co-ordinates gives

$$\tilde{P} = \frac{\beta y F}{2\pi\sigma^2}\exp\left[i(\omega t + \epsilon M\bar{x})\right] \tag{8}$$

for the approximate (denoted by the tilde on P) solution for a line force. If the exact solution for the pressure, equation (3), is expanded for small ϵ, it is found to differ from (8) only in terms of $O(\epsilon^2)$ and higher, as might be expected since the term neglected in the wave equation in deriving (8) was $O(\epsilon^2)$. The velocity $\tilde{v}$ under this approximation is found by introducing (8) into (4). The result is given by (5) with I_1 replaced by

$$\tilde{I}_1 = \frac{1}{2\pi}\left\{-\frac{1}{\bar{x}}\exp(ik^*\bar{x}) + ik^*\left[\gamma + \ln(k^*|\bar{x}|) + \tfrac{1}{2}i\pi - ik^*\bar{x}\int_0^1 \ln\xi\exp(ik^*\bar{x}\xi)\,d\xi\right]\right\}, \tag{9}$$

where $\gamma = 0{\cdot}577\ldots$ is Euler's constant.

To determine the order of accuracy of the approximate velocity $\tilde{v}$, $\tilde{I}_1$ will be compared with an expansion of I_1 for small ϵ. Expanding the Hankel functions in (6) for small ϵ and ignoring terms $O(\epsilon^2)$ in the equation gives

$$I_1 = \tilde{I}_1 + (ik^*/2\pi)f(M) + O(\epsilon^2), \tag{10}$$

where

$$f(M) \equiv (1-\beta)\ln M + \beta\ln(1+\beta) - \ln 2 \tag{11}$$

is the function of Mach number introduced by Miles (1950*b*). Thus, even though the $O(\epsilon)$ terms in the pressure expressions P and $\tilde{P}$ are identical, the $O(\epsilon)$ terms in the velocities v and $\tilde{v}$ differ. The conclusion is that ignoring the P_{TT} term in the transformed wave equation will give incorrect $O(\epsilon)$ results for the two-dimensional problem with shed vorticity, since a boundary condition on the velocity must be satisfied.

Two interesting additional points will be mentioned before passing to the three-dimensional case. It will be noted that the expansion leading to (10) made no assumption that the Mach number be real. The solution of the Possio integral equation for imaginary values of the Mach number is a relevant case discussed by Graham (1970). Since (10) [and also (12)] holds for imaginary M, the approximate solution to the Possio integral equation given by Amiet (1974) can be shown to retain its validity for imaginary M. This fact is applied by Amiet (1976*b*) to obtain an approximate solution for the problem of an infinite-span airfoil encountering a skewed gust in incompressible flow.

Also, (10) can be rewritten, incurring an error $O(\epsilon^2)$, as

$$I_1 = \tilde{I}_1 \exp[-ik^*\bar{x}f(M)] + O(\epsilon^2). \tag{12}$$

Since (5) is the kernel function for the Possio integral equation, writing the approximation for I_1 in the form (12) allows a simple inversion to obtain an approximate solution; i.e. by redefining the unknown in the integrand of the Possio integral equation to include the factor $\exp[-ik^*\bar{x}(f(M)-M^2)]$ and noting that $\tilde{I}_1 \exp(-ik^*\bar{x})$ is the kernel function for incompressible flow (but with the reduced frequency k replaced by k^*), the approximate solution given by Amiet (1974) can be derived by direct analogy with the incompressible solution. A somewhat different method of deriving this approximate solution, beginning with an equation similar to (10) rather than (12), but still using the analogy with the incompressible solution, was given by Kemp & Homicz (1976).

3. Three-dimensional case with shed vorticity

If the line force of the preceding case is replaced by a point force

$$\mathbf{F} = \mathbf{j}F\,\delta(x)\,\delta(y)\,\delta(z)\,e^{i\omega t} \tag{13}$$

the results corresponding to those for the two-dimensional case are

$$P = \frac{F}{4\pi}\exp[i(\omega t + \epsilon M\bar{x})]\frac{\partial}{\partial y}\left(\frac{1}{\sigma}e^{-i\epsilon\bar{\sigma}}\right), \tag{14a}$$

$$\tilde{P} = \frac{F}{4\pi}\exp[i(\omega t + \epsilon M\bar{x})]\frac{\partial}{\partial y}\left(\frac{1}{\sigma}\right) = P + O(\epsilon^2). \tag{14b}$$

The velocity on the axis ahead of the dipole ($x < 0$) is

$$v(x<0,0,0) = -\frac{\beta^2 F I_2}{8\pi\rho_0 U b^2}\exp[i(\omega t - k\bar{x})], \tag{15}$$

where

$$I_2 = \frac{1}{\bar{x}^2}[1 + i\bar{x}(k^* - \epsilon)]\exp[i\bar{x}(k^* + \epsilon)] - k^{*2}\beta^2\int_1^\infty \exp[i\bar{x}(k^*+\epsilon)\xi]\frac{d\xi}{\xi}, \tag{16a}$$

$$\tilde{I}_2 = \frac{1}{\bar{x}^2}(1 + ik^*\bar{x})\exp(ik^*\bar{x}) - k^{*2}\int_1^\infty \exp(ik^*\bar{x}\xi)\frac{d\xi}{\xi}. \tag{16b}$$

Using the relation

$$\int_1^\infty e^{-ik\xi}\frac{d\xi}{\xi} = -\gamma - \ln k - i\frac{\pi}{2} - \int_0^1 (e^{-ik\xi}-1)\frac{d\xi}{\xi} \tag{17}$$

in the integrals in (16) and expanding for small ϵ gives

$$I_2 = \tilde{I}_2 + O(\epsilon^2). \tag{18}$$

Thus the exact and the approximate solutions for both the pressure and the velocity differ by terms $O(\epsilon^2)$, showing that ignoring the $O(\epsilon^2)$ term P_{TT} in the transformed wave equation leads to the correct term $O(\epsilon)$ for the three-dimensional problem with shed vorticity.

4. Two- and three-dimensional cases without shed vorticity

The absence of shed vorticity simplifies the problem considerably since there is then no discontinuity in the velocity potential in the wake. As discussed by Lamb (1932) the velocity potential can then be represented by a distribution of monopole and dipole sources over the body, whereas if shed vorticity were present the pressure field could be represented by such a distribution, but not the velocity potential.

When shed vorticity is absent, the combination of Galilean and Lorentz transforms given by (7) reduces the problem to an equivalent zero-flow problem. (If shed vorticity is present, this transformation would bring both the body and the fluid to rest, but it would not eliminate the required vorticity distribution downstream of the body.) Thus, for problems without shed vorticity, only the zero-flow case need be considered. For zero flow the parameter ϵ reduces to $\omega b/c_0$.

The velocity potential ϕ for the zero-flow case can be written as (Lamb 1932, pp. 498, 531)

$$\phi = \iint \left[-g(r)\frac{\partial\phi}{\partial n} + \phi\frac{\partial g(r)}{\partial n}\right] dS, \tag{19}$$

which is a distribution of monopole and dipole sources over the body surface with the strengths of the sources being $\partial\phi/\partial n$ and ϕ. The normal n points outwards from the body and r is the distance between the source and observer normalized by a typical body dimension b. For two-dimensional problems

$$g(r) = -\tfrac{1}{4}iH_0^{(2)}(\epsilon r) \tag{20}$$

and for three-dimensional problems

$$g(r) = (4\pi r)^{-1}e^{-i\epsilon r}. \tag{21}$$

For the two-dimensional problem, introducing (20) into (19) and expanding for small ϵ gives

$$\phi = \frac{i}{4}\int\left\{\left[1 - \frac{2i}{\pi}\left(\ln\frac{\epsilon r}{2}+\gamma\right)\right]\frac{\partial\phi}{\partial n} + \frac{2i}{\pi}\phi\frac{\partial}{\partial n}\ln r\right\} dS + O(\epsilon^2). \tag{22}$$

On taking the gradient to find the velocity, the terms $O(\epsilon)$ drop out (since the function ϕ under the integral is a function only of the integration variables) with the result that

$$\mathbf{v} = \frac{1}{2\pi}\nabla\int\left[\frac{\partial\phi}{\partial n}\ln r - \phi\frac{\partial}{\partial n}\ln r\right]dS + O(\epsilon^2). \tag{23}$$

This is the counterpart, for problems without shed vorticity, of the Possio integral equation. There is no $O(\epsilon)$ term, and the equation is exactly that which would be obtained by ignoring the P_{TT} term in the wave equation.

For three-dimensional problems the corresponding results are

$$\phi = \frac{1}{4\pi}\iint\left[\left(-\frac{1}{r}+i\epsilon\right)\frac{\partial\phi}{\partial n} + \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS + O(\epsilon^2), \tag{24}$$

$$\mathbf{v} = \frac{1}{4\pi}\nabla\iint\left[-\frac{1}{r}\frac{\partial\phi}{\partial n} + \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS + O(\epsilon^2). \tag{25}$$

The conclusion is again that ignoring the $O(\epsilon^2)$ term in the wave equation leads to correct $O(\epsilon)$ results.

In conclusion, the $O(\epsilon)$ term for two-dimensional problems without shed vorticity and general three-dimensional problems can be obtained by ignoring the P_{TT} term in the transformed wave equation. For two-dimensional problems with shed vorticity downstream, the Possio integral equation or its equivalent must be used to obtain the $O(\epsilon)$ term.

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